

A CUBICAL FLAT TORUS THEOREM AND THE BOUNDED PACKING PROPERTY

DANIEL T. WISE AND DANIEL J. WOODHOUSE

ABSTRACT. We prove the bounded packing property for any abelian subgroup of a group acting properly and cocompactly on a $\text{CAT}(0)$ cube complex. A main ingredient of the proof is a cubical flat torus theorem. This ingredient is also used to show that central HNN extensions of maximal free-abelian subgroups of compact special groups are virtually special, and to produce various examples of groups that are not cocompactly cubulated.

1. INTRODUCTION

Let G be a finitely generated group, and let Υ be its Cayley graph with respect to some finite generating set. A subgroup $H \leq G$ has *bounded packing in G* if for each $r > 0$ there exists $m = m(r)$ such that if g_1H, \dots, g_mH are distinct left cosets of H , then there exists i, j such that $d_\Upsilon(g_ih, g_jh') > r$ for all $h, h' \in H$.

The motivating goal of this article is to prove the following:

Theorem 3.7. *Let G act properly and cocompactly on a $\text{CAT}(0)$ cube complex \tilde{X} . Let A be an abelian subgroup of G . Then A has bounded packing in G .*

Since Theorem 3.7 is limited to the setting of $\text{CAT}(0)$ cube complexes, it offers no direction towards resolving the following problems:

Problem 1.1.

- (1) Let G act properly and cocompactly on a $\text{CAT}(0)$ space. Does each [cyclic] abelian subgroup $A \leq G$ have bounded packing?
- (2) Let G be an aTmenable group. Does each abelian subgroup $A \leq G$ have bounded packing?
- (3) Find a finitely generated group G with an infinite cyclic subgroup $A \leq G$ that does not have bounded packing.

The *rank* of a virtually abelian group A is the rank of any finite index free-abelian subgroup of A . A virtually abelian subgroup $A \leq G$ is *highest* if A does not have a finite index subgroup that lies in a virtually abelian subgroup of higher rank. The particular feature of $\text{CAT}(0)$ cube complexes used to prove Theorem 3.7 is Theorem 2.1, which is the crux of this paper. A neat consequence of it is the

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following Cubical Flat Torus Theorem which asserts that a highest abelian subgroup acts on a product of quasilines. A *cubical quasiline* is a $\text{CAT}(0)$ cube complex that is quasi-isometric to \mathbb{R} . For brevity we will simply refer to cubical quasilines as *quasilines*.

Theorem 3.6. *Let G act properly and cocompactly on a $\text{CAT}(0)$ cube complex \tilde{X} . Let A be a highest virtually abelian subgroup of G and let $p = \text{rank}(A)$. Then A acts properly and cocompactly on a convex subcomplex $\tilde{Y} \subseteq \tilde{X}$ such that $\tilde{Y} \cong \prod_{i=1}^p C_i$ where each C_i is a quasiline.*

We also present the following application of Theorem 3.7:

Theorem 5.5. *Let H be a finitely generated virtually [compact] special group. Let $A \subset H$ be a highest abelian subgroup. Let $G = H *_{A^t=A}$ be the HNN extension, where t is the stable letter commuting with A , then G is virtually [compact] special.*

A version of Theorem 5.5 was proven in [Wis] under the additional hypothesis of relative hyperbolicity, but Theorem 3.7 allows us to avoid this hypothesis. Example 4.5 shows that G can fail to have a virtually compact cubulation when H is a f.g. 2-dimensional right-angled Artin group, but A is not highest.

Section 4 uses Theorem 3.6 to restrict how highest abelian subgroups intersect. The following amusing consequence of Corollary 4.4 shows that generic multiple cyclic HNN extensions of \mathbb{Z}^p cannot be virtually compactly cubulated:

Example 4.6. *Let $\{\langle b_1 \rangle, \dots, \langle b_r \rangle, \langle c_1 \rangle, \dots, \langle c_r \rangle\}$ be a collection of pairwise incommensurable infinite cyclic subgroups of \mathbb{Z}^p , and suppose that $r > \frac{p}{2}$. Let G be the following multiple HNN extension of $\mathbb{Z}^p = \langle a_1, \dots, a_p \rangle$:*

$$G = \langle a_1, \dots, a_p, t_1, \dots, t_r \mid [a_i, a_j] = 1, b_k^{t_k} = c_k : 1 \leq k \leq r \rangle$$

Then G does not contain a finite index subgroup that acts properly and cocompactly on a $\text{CAT}(0)$ cube complex.

This paper is structured as follows: In Section 2 we prove Theorem 2.1. In Section 3 we collect existing results and explain how, along with Theorem 2.1, they allow us to prove Theorem 3.7. In Section 4 we observe that highest free-abelian subgroups have restricted intersections with other free-abelian subgroups. In Section 5 we prove Theorem 5.5.

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2. THE DUAL TO A FLAT

The goal of this section is to prove Theorem 2.1, which we state below. A *quasiline* is a $\text{CAT}(0)$ cube complex quasi-isometric to \mathbb{R} , and a *quasiray* is a $\text{CAT}(0)$ cube complex quasi-isometric to $[0, \infty) \subseteq \mathbb{R}$. A hyperplane in a $\text{CAT}(0)$ cube complex \tilde{X} will be denoted by H , and its left and right halfspaces are denoted by \overleftarrow{H} and \overrightarrow{H} . As it is convenient to work with subcomplexes, we define $\overleftarrow{\tilde{H}}, \overrightarrow{\tilde{H}}$ to be the smallest subcomplexes containing the complementary components of $\tilde{X} - H$. We will be

using wallspaces and Sageev's dual cube complex construction [Sag95]. We point the reader to [HW] for an account of the techniques. The proof of the following is given at the end of this section, after we have developed the required language and lemmas.

A subset $S \subseteq \tilde{X}$ of a geodesic metric space is *convex* if every geodesic with endpoints in S is contained in S . When \tilde{X} is a complete CAT(0) space, a complete connected subspace Y is convex if its inclusion into \tilde{X} is a local isometry. When \tilde{X} is a CAT(0) cube complex, and Y is a subcomplex, there is a simple combinatorial criterion equivalent to being a local isometry: For each 0-cube v of Y the inclusion $\text{link}_Y(v) \hookrightarrow \text{link}_{\tilde{X}}(v)$ is a full subcomplex. We refer to [BH99] for a comprehensive account of CAT(0) metric spaces.

For a subset S of a CAT(0) cube complex \tilde{X} , let $\text{hull}(S)$ be the smallest nonempty convex subcomplex of \tilde{X} that contains S .

Theorem 2.1. *Let $A \leq G$ be a virtually abelian subgroup of rank p that acts properly and cocompactly on a flat E in a CAT(0) cube complex \tilde{X} . Then either:*

- (1) *$\text{hull}(E)$ is A -cocompact and $\text{hull}(E) \cong \prod_{i=1}^p C_i$, where each C_i is a convex subcomplex that is a quasiline.*
- (2) *There exists a finite index subgroup $B \leq A$ such that $\min(B) \cap \text{hull}(E)$ is not B -cocompact.*

Example 2.2. Consider the cyclic group A generated by a diagonal glide reflection acting on the standard cubulation of the plane \mathbb{R}^2 . Then $\min(A)$ is a diagonal line while $\text{hull}(E)$ is \mathbb{R}^2 .

Let $A \leq G$ be a virtually abelian subgroup of rank p that acts properly and cocompactly on a flat E in a CAT(0) cube complex \tilde{X} . By a result of Bieberbach [Rat94], there exists a finite index free-abelian subgroup $A_t \leq A$ that acts by translations on E . Let P be the set of all hyperplanes intersecting E . The hyperplanes H_1, H_2 are *parallel in E* if $H_1 \cap E$ and $H_2 \cap E$ are parallel in E . Being parallel in E is an equivalence relation on the hyperplanes intersecting E . There are finitely many parallelism classes of hyperplanes in E , denoted $P_i \subseteq P$ for $1 \leq i \leq p$.

Lemma 2.3. *There exists a finite index subgroup $B \leq A_t$ that acts disjointly in the sense that distinct hyperplanes in the same B -orbit are disjoint.*

Proof. For each parallelism class P_i , choose $g_i \in A_t$ such that the axis of g_i crosses $H \cap E$ for $H \in P_i$. There exists $n_i > 0$ such that $\langle g_i^{n_i} \rangle$ acts disjointly on the hyperplanes in P_i , as otherwise $g_i^j H$ intersects $g_i^k H$ for all $j, k \in \mathbb{Z}$, contradicting that $\dim(\tilde{X}) < \infty$. Indeed m pairwise intersecting hyperplanes mutually intersect in an m -cube. As a CAT(0) cube complex is dual to the wallspace associated to its collection of hyperplanes, a point in \tilde{X} , together with such a collection of pairwise crossing hyperplanes determines an m -cube (see eg [HW]). Alternatively, one can reach a contradiction from Proposition 3.2 applied to the subdivision of \tilde{X} .

For $\epsilon > 0$ to be determined below, we let E^ϵ denote $\mathcal{N}_\epsilon(E)$ and for a hyperplane $H \in P_i$ we let $H^\epsilon = H \cap E^\epsilon$ and let $P_i^\epsilon = \{H^\epsilon : H \in P_i\}$. Thus $(E^\epsilon, P_i^\epsilon)$ is a wallspace for each i . We choose ϵ so that for each i , for each pair $H, H' \in P_i$ we

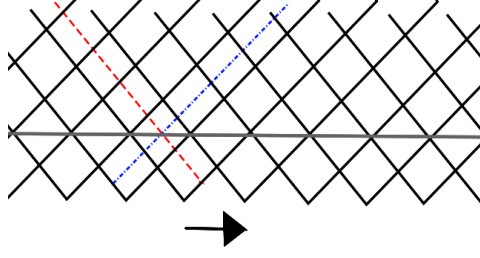


FIGURE 1. A cubical halfplane with two semi-crossing hyperplane orbits.

have $H \cap E = H' \cap E$ if and only if H, H' cross within E^ϵ . Cocompactness of E ensures that there are finitely many intersection angles between hyperplanes and E , and this allows us to bound the number of orbits of codimension-2 hyperplanes intersecting $\mathcal{N}_1(E)$ but not intersecting E . We choose ϵ to be less than the minimal distance from E to any such codimension-2 hyperplane.

The dual cube complex $C(E^\epsilon, P_i^\epsilon)$ is a quasiline with an A_t -action. The kernel K_i of this action is isomorphic to \mathbb{Z}^{p-1} . Adjoining $g_i^{n_i}$ to K_i , we obtain the finite index subgroup $B_i = \langle K_i, g_i^{n_i} \rangle \leq A_t$. For each $H \in P_i$ we have $B_i H = \langle g_i^{n_i} \rangle H$ and hence B_i acts disjointly on hyperplanes in P_i . Finally, $B = \bigcap_i B_i$ acts disjointly on the hyperplanes in P . \square

For each parallelism class P_i , let $Z_i \leq B$ be an infinite cyclic group stabilizing a line $R_i \subseteq E$, non parallel to $H \cap E$ for all $H \in P_i$. Consider two hyperplanes $H, H' \in P_i$ with distinct Z_i -orbits: $\{z_i^j H : j \in \mathbb{Z}\}$ and $\{z_i^j H' : j \in \mathbb{Z}\}$. Observe that if H intersects both $z_i^j H'$ and $z_i^k H'$, then H intersects $z_i^\ell H'$ for $j \leq \ell \leq k$.

Definition 2.4. We say $H, H' \in P$ represent *crossing* orbits if either $H \in P_i$ and $H' \in P_j$ with $i \neq j$, or if $H, H' \in P_i$ and $z_i^j H$ crosses $z_i^k H'$ for all $j, k \in \mathbb{Z}$. We say $H, H' \in P_i$ represent *aligned* orbits if H intersects only finitely many Z_i -translates of H' . We say $H, H' \in P_i$ represent *semi-crossing* orbits if they do not represent crossing orbits, and there exists $m \in \mathbb{Z}$ such that either $z_i^j H$ crosses $z_i^k H'$ for $j - k > m$, or $z_i^j H$ crosses $z_i^k H'$ for $k - j > m$. Any pair of hyperplanes in P must represent either crossing, semicrossing, or aligned orbits.

Example 2.5. Let \mathbb{Z} act on the cubical halfplane in Figure 1. The flat plane E is the line homeomorphic to \mathbb{R} . Observe that there are two parallelism classes of orbits that semi-cross. However, the action is not cocompact.

Lemma 2.6. *Alignment of Z_i -orbits is an equivalence relation.*

Proof. Reflexivity and symmetry are immediate. Suppose $Z_i H$ is aligned to $Z_i H'$ and $Z_i H'$ is aligned to $Z_i H''$, but infinitely many hyperplanes in $Z_i H$ cross H'' . By their alignment, there exists j, k such that $H'' \subset (z_i^j \overrightarrow{H'} \cap z_i^k \overleftarrow{H'})$. Since infinitely many elements of $Z_i H$ intersect H'' , infinitely many of these elements intersect H'' but do not intersect $z_i^j H'$ or $z_i^k H'$. This is a contradiction as infinitely many hyperplanes cannot separate $z_i^j H'$ and $z_i^k H'$. \square

Lemma 2.7. *Semi-crossing of Z_i -orbits is a partial ordering, denoted by $>$, where $Z_i H > Z_i H'$ if $Z_i H$ and $Z_i H'$ are semi-crossing and there exists $m \in \mathbb{Z}$ such that $z_i^j H$ crosses $z_i^k H'$ for $j - k > m$.*

Proof. Antisymmetry holds since, if $Z_i H > Z_i H'$ and $Z_i H' > Z_i H$ but $Z_i H \neq Z_i H'$ then $Z_i H$ and $Z_i H'$ are crossing orbits. However, distinct crossing orbits are not semi-crossing.

To prove transitivity we first prove the following claim: If $Z_i H_1$ and $Z_i H_2$ are aligned orbits and $Z_i H' > Z_i H_1$, then $Z_i H' > Z_i H_2$. By their alignment, there exists $p < q$ such that $H_2 \subset (z_i^p \vec{H}_1 \cap z_i^q \vec{H}_1)$. Suppose $z_i^j H'$ crosses $z_i^k H_1$ precisely for $j - k > m$. Then H_2 is crossed by $z_i^j H'$ for $j - q > m$ as $z_i^j H'$ crosses both $z_i^q H_1$ and $z_i^p H_1$. Similarly H_2 is not crossed by $z_i^j H'$ for $j - p \leq m$. This implies that $Z_i H' > Z_i H_2$. Similarly $Z_i H' < Z_i H_1$ would imply $Z_i H' < Z_i H_2$.

Suppose that $Z_i H_3 > Z_i H_2 > Z_i H_1$. By the claim, $Z_i H_1$ cannot be aligned to $Z_i H_3$. Therefore we need only exclude the possibility that $Z_i H_1$ and $Z_i H_3$ are crossing orbits. Since $Z_i H_2 > Z_i H_1$ there exists N_1 such that $z_i^n H_2$ is disjoint from H_1 for all $n \leq N_1$. Assume that $H \subseteq z_i^{N_1} \vec{H}_2$. Since $Z_i H_3 > Z_i H_2$ there exists N_2 such that $z_i^n H_3$ is disjoint from $z_i^{N_1} H_2$ for all $n \leq N_2$. Since z_i acts by translation on E we can deduce that $z_i^n H_3 \subseteq z_i^{N_1} \vec{H}_2$ for $n \leq N_2$. Hence, $z_i^n H_3$ is disjoint from H_1 for all $n \leq N_2$ as they are separated by $z_i^{N_1} H_2$. \square

Proof of Theorem 2.1. We first assume there are no semi-crossing orbits. In the next three steps we will show that $\text{hull}(E) = \prod_{i=1}^p C_i$ where each C_i is the quasiline dual to the family of hyperplanes corresponding to an alignment class.

First, we claim that $\text{hull}(E)$ is isomorphic to $C(\tilde{X}, P)$. Indeed, each 0-cube x in $\text{hull}(E)$ corresponds to the 0-cube y in $C(\tilde{X}, P)$ where each hyperplane in P is oriented towards x . Conversely, a 0-cube y in $C(\tilde{X}, P)$ determines a 0-cube x in \tilde{X} by orienting the hyperplanes not in P towards E . Note x lies in $\text{hull}(E)$ since each $x[H] \cap E \neq \emptyset$ for all hyperplanes H . This bijection preserves adjacency.

Secondly, let $\{A_i\}_{i=1}^p$ be an enumeration of the alignment classes. Observe $C(\tilde{X}, P) \cong \prod_{i=1}^p C(\tilde{X}, A_i)$ as every hyperplane in A_i intersects every hyperplane in A_j for $i \neq j$. Indeed, a 0-cube of $C(\tilde{X}, P)$ determines a 0-cube of each of the factors by ignoring the orientations in the other alignment classes. Conversely, a choice 0-cubes in each of the factors determines a 0-cube in $C(\tilde{X}, P)$ since the hyperplanes cross each other. Again, it is easy to see this bijection preserves adjacency.

Thirdly, let $G_i = \langle g_i \rangle$ be an infinite cyclic subgroup of B acting freely on $C(\tilde{X}, A_i)$. We will show that $C(\tilde{X}, A_i)$ is G_i -cocompact, and therefore quasiisometric to \mathbb{R} . Let H_1, \dots, H_k be representatives of the distinct G_i -orbits. Note that the dimension of $C(\tilde{X}, A_i)$ is bounded by k . We now show that there are finitely many G_i -orbits of maximal cubes. A maximal cube corresponds to a collection of pairwise intersecting hyperplanes $g_i^{\alpha_1} H_{j_1}, \dots, g_i^{\alpha_\ell} H_{j_\ell}$. By translating we can assume that $\alpha_1 = 0$, and therefore there are finitely many such collections since only finitely many hyperplanes can intersect H_{j_1} .

Then $\text{hull}(E) = \prod_{i=1}^p C_i$ where each C_i is the quasiline dual to the family of hyperplanes corresponding to an alignment class. Observe that B acts by translations on E with disjoint hyperplane-orbits and hence stabilizes each alignment class and thus preserves the factors of the product structure. If $p = \text{rank}(A)$ the action on $\text{hull}(E)$ is cocompact, which implies that the set of hyperplanes orthogonal to each R_i belong to a single alignment class. Otherwise $p > \text{rank}(A)$, and as B acts metrically properly and cocompactly on C_i , each C_i contains an isometrically embedded B -invariant line ℓ_i . Thus $\prod \ell_i \subseteq \prod_{i=1}^p C_i$ is not cocompact, but is contained in $\min(B) \cap \text{hull}(E)$.

Suppose there are at least two semi-crossing orbits in some parallelism class, and let Q be a maximal alignment class with respect to the partial ordering. For each parallelism class P_i and orbit $Z_i H \subseteq P_i - Q$: either $Z_i H$ crosses the orbits in Q , or $Q \subset P_i$ and $Z_i H' > Z_i H$ for all $H' \in Q$.

We define a sequence of B -equivariant cubical maps $\{\phi_k : \text{hull}(E) \rightarrow \text{hull}(E)\}_{k \in \mathbb{N}}$ using the partition $P = Q \sqcup Q^c$: A 0-cell x in $\text{hull}(E)$ corresponds uniquely to a choice of orientation for each hyperplane intersecting E . Let $x[H] \in \{\overleftarrow{H}, \overrightarrow{H}\}$ denote the halfspace of H containing x in its interior. Its image $\phi_k(x)$ is specified by how $\phi_k(x)$ orients the hyperplanes intersecting E . For $H \in Q^c$ let $\phi_k(x)[H] = x[H]$. For $H \in Q \subseteq P_i$ let $\phi_k(x)[z_i^k H] = x[H]$. This defines a 0-cube in $\text{hull}(E)$ since only finitely many hyperplanes have their orientations changed, and disjoint hyperplanes are not oriented away from each other by $\phi_k(x)$: Let $H \subseteq P_i \subseteq Q^c$ represent a Z_i -orbit not crossing the Z_i -orbits in $Q \subseteq P_i$, then $Z_i H' > Z_i H$ for any $H' \in Q$. Therefore, if H' crosses H then $z_i^k H'$ also crosses H .

The injectivity of ϕ_k on 0-cubes holds since if $x_1 \neq x_2$ then there exists $H \in P$ such that $x_1[H] \neq x_2[H]$. If $H \in Q^c$ then $\phi_k(x_1)[H] = x_1[H] \neq x_2[H] = \phi_k(x_2)[H]$ so $\phi_k(x_1) \neq \phi_k(x_2)$. If $H \in Q$ then $\phi_k(x_1)[z_i^{-k} H] = x_1[H] \neq x_2[H] = \phi_k(x_2)[z_i^{-k} H]$ so $\phi_k(x_1) \neq \phi_k(x_2)$. Therefore ϕ_k is injective on the 0-skeleton. Similar reasoning shows that ϕ_k sends adjacent 0-cubes to adjacent 0-cubes and so ϕ_k extends to the 1-skeleton of $\text{hull}(E)$. Moreover injective maps on the 1-skeleton send squares to squares, hence the map also extends to the 2-skeleton.

Any map defined on the 2-skeleton of a cube complex extends uniquely to a cubical map on the entire complex. Observe that B acts on E by translation and preserves each Z_j -orbit in each P_j . Therefore, for each $b \in B$ there exists ℓ_i , for $1 \leq i \leq p$, such that $bH = z_i^{\ell_i} H$ for each $H \in P_i$. Therefore ϕ_k is B -equivariant since if $H \in P_i$ but $H \notin Q$ then

$$\begin{aligned} (b \cdot \phi_k(x))[H] &= \phi_k(x)[b^{-1} H] = \phi_k(x)[z_i^{-\ell_i} H] \\ &= x[z_i^{-\ell_i} H] = x[b^{-1} H] = (b \cdot x)[H] = \phi_k(b \cdot x)[H]. \end{aligned}$$

Similarly, if $H \in Q \subseteq P_i$ then

$$\begin{aligned} (b \cdot \phi_k(x))[H] &= \phi_k(x)[b^{-1} H] = \phi_k(x)[z_i^{-\ell_i} H] \\ &= x[z_i^{k-\ell_i} H] = x[b^{-1} z_i^k H] = (b \cdot x)[z_i^k H] = \phi_k(b \cdot x)[H]. \end{aligned}$$

We now show that $d(\phi_k(x), b \cdot x) \geq k$ for each $b \in B$ and x a canonical 0-cube x associated to a point in E . For each Z_j -orbit $Z_j H$ fix a representative

H such that $x \in \vec{H} \cap z_i \overleftarrow{H}$. Let $H \in Q \subseteq P_i$ be such a representative, then ϕ_k changes the orientation of precisely k hyperplanes in $Z_i H$, namely $z_i H, \dots, z_i^k H$. For any representative $H \in P_i$, however, translation by b changes the orientation of ℓ_i hyperplanes, namely $z_i H, \dots, z_i^{\ell_i} H$. As there is at least one Z_i -orbit in $P_i \supseteq Q$ not in Q , we can deduce that at least k hyperplanes have distinct orientations in $\phi_k(x)$ and $b \cdot x$. Therefore the distance from bx to $\phi_k(x)$ is at least k .

Observe that $d(\phi_k(y_1), \phi_k(y_2)) \leq d(y_1, y_2)$ for $y_1, y_2 \in \text{hull}(E)$. Indeed, the CAT(0) metric on $\text{hull}(E)$ is defined to be the infimal length of piecewise Euclidean paths joining points, and the map preserves lengths of paths. The B -equivariance together with that ϕ_k is distance-nonincreasing implies that $\phi_k(e) \in \min(B)$ for each $e \in E$.

In conclusion, $d(\phi_k(E), E) \rightarrow \infty$ as $k \rightarrow \infty$. For $e \in E$, the orbit $Be \subseteq E$ is mapped by a distance-nonincreasing function to a new orbit at distance $\geq k$ from E . Since the original flat was in $\min(B)$, the image of the image orbit is isometric to the original orbit. Since k is unbounded, $\min(B) \cap \text{hull}(E)$ is non-cocompact. \square

3. THE BOUNDED PACKING PROPERTY

Let G be a finitely generated group with Cayley graph Υ . Suppose G acts by isometries on a geodesic metric space \tilde{X} such that the map $g \mapsto gx_0$ is a quasi-isometric embedding for some $x_0 \in \tilde{X}$. Then $H \leq G$ has bounded packing if and only if for each $r > 0$ there exists $m = m(r)$ such that if $g_1 H, \dots, g_m H$ are distinct left cosets of H , then there exists i, j such that $d_{\tilde{X}}(g_i H x_0, g_j H x_0) > r$. We refer to [HW09] for more about bounded packing, and specifically to Cor 2.9, Lem 2.3, and Lem 2.4 for the following:

Lemma 3.1. *Suppose H has bounded packing in G . If $K \leq H$ is a normal subgroup of H , then K has bounded packing in G . If $K \leq G$ is a subgroup such that $[H : K \cap H] < \infty$ and $[K : K \cap H] < \infty$, then K has bounded packing.*

A proof of the following well known fact follows from the median space structure of the 1-skeleton of a CAT(0) cube complex [Rol98, Thm 2.2]. A proof using disk diagrams can be found in [Wis, Sec 2].

Proposition 3.2 (Helly Property). *Let Y_1, \dots, Y_r be convex subcomplexes of a CAT(0) cube complex. If $Y_i \cap Y_j \neq \emptyset$ for each i, j , then $\cap_{i=1}^r Y_i \neq \emptyset$.*

The following is obtained in [HW08, Lem 13.15]:

Lemma 3.3. *Let $\tilde{Y} \subset \tilde{X}$ be a convex subcomplex of the CAT(0) cube complex \tilde{X} . For each $r \geq 0$ there exists a convex subcomplex \tilde{Y}^{+r} such that $\mathcal{N}_r(\tilde{Y}) \subset \tilde{Y}^{+r} \subset \mathcal{N}_s(\tilde{Y})$ for some $s \geq 0$. Here $\mathcal{N}_m(\tilde{Y})$ denotes the m -neighborhood of \tilde{Y} .*

We infer the following from the above results.

Lemma 3.4. *Let G act properly and cocompactly on a CAT(0) cube complex \tilde{X} . Let H be a subgroup that cocompactly stabilizes a nonempty convex subcomplex $\tilde{Y} \subset \tilde{X}$. Then H has bounded packing in G .*

Proof. Let $x_o \in \tilde{Y}$. Let g_1, g_2, \dots be an enumeration of the left coset representatives of H . Let \tilde{Y}^{+r} be as in Lemma 3.3. Observe that if $d(g_j H x_o, g_k H x_o) < r$ then $g_j \tilde{Y}^{+r} \cap g_k \tilde{Y}^{+r} \neq \emptyset$. Thus, to show that H has bounded packing, it suffices to find an upper bound on the number of distinct cosets $g_i H$ such that $\{g_i \tilde{Y}^{+r}\}$ pairwise intersect. Moreover, by Proposition 3.2, if $\{g_1 \tilde{Y}^{+r}, \dots, g_p \tilde{Y}^{+r}\}$ pairwise intersect then $\cap_{i=1}^p g_i \tilde{Y}^{+r} \neq \emptyset$. It thus suffices to show that there is an upper bound m on the multiplicity of $\{g_i \tilde{Y}^{+r} : g_i H \in G/H\}$. However this collection of sets is uniformly locally finite since \tilde{Y}^{+r} is H -cocompact and $[\text{Stabilizer}(Y^{+r}) : H] < \infty$. \square

If A is an abelian group acting by isometries on a metric space \tilde{X} , then $\min(A)$ is the set of all $\tilde{x} \in \tilde{X}$ such that $d(a\tilde{x}, \tilde{x}) \leq d(a\tilde{y}, \tilde{y})$ for all $a \in A$ and $\tilde{y} \in \tilde{X}$. We refer to [BH99] for the following:

Proposition 3.5 (Flat torus theorem). *Let A be a virtually free-abelian group of rank n acting metrically properly and semisimply on a $\text{CAT}(0)$ space \tilde{X} . There exists a subspace $V \times F \subset \tilde{X}$ with F isometric to \mathbb{E}^n such that A stabilizes $V \times F$ and acts as: $a(v, f) = (v, af)$ for all $(v, f) \in V \times F$ and $a \in A$. Moreover, if $A \cong \mathbb{Z}^n$ then $\min(A) = V \times F$.*

Theorem 3.6 (Cubical flat torus theorem). *Let G act properly and cocompactly on a $\text{CAT}(0)$ cube complex \tilde{X} . Let A be a highest virtually abelian subgroup of G and let $p = \text{rank}(A)$. Then A acts properly and cocompactly on a convex subcomplex $\tilde{Y} \subseteq \tilde{X}$ such that $\tilde{Y} \cong \prod_{i=1}^p C_i$ where each C_i is a quasiline.*

Proof. By Proposition 3.5, A stabilizes $E \subset \tilde{X}$, where E is isometric to \mathbb{E}^p . By Theorem 2.1, either $\text{hull}(E) \cong \prod_{i=1}^p C_i$ is A -cocompact where each C_i is a quasiline, or there exists a finite index free-abelian subgroup $B \leq A$ such that $\min(B) \cap \text{hull}(E)$ is not B -cocompact. We shall show that the second possibility contradicts that A is highest.

Applying Proposition 3.5 again, let $\min(B) = V \times F$, where $\text{diam}(V) = \infty$. For $v \in V$ let $N(\{v\} \times F)$ denote the smallest B -invariant connected subcomplex of \tilde{X} containing $\{v\} \times F$. Since $\{v\} \times F$ is B -cocompact, so is $N(\{v\} \times F)$. Moreover, the number of B -orbits of cells in $N(\{v\} \times F)$ is bounded by a constant independent of $v \in V$. Indeed, by B -cocompactness, there is $m > 0$ such that $F = B\mathcal{N}_m(f)$ for each $f \in F$. For each v there is a B -equivariant isometry $F \rightarrow \{v\} \times F$, and so $\{v\} \times F$ is likewise covered by the B translates of each m -ball. However, the number of cells intersecting an m -ball in \tilde{X} is finite by properness and cocompactness. So the number of B -orbits of cells in $N(\{v\} \times F)$ has the same upper bound.

It follows that there are finitely many G -orbits of subcomplexes $N(\{v\} \times F)$. As $\text{diam}(V) = \infty$, there are points $v_1, v_2 \in V$ and $g \in G$ such that $N(\{v_1\} \times F) \cap N(\{v_2\} \times F) = \emptyset$, but $gN(\{v_1\} \times F) = N(\{v_2\} \times F)$. Both B and g stabilize $\sqcup g^n N(\{v_1\} \times F)$ which is quasi-isometric to \mathbb{E}^{n+1} . Hence $\langle g, B \rangle$ is a higher rank virtually abelian subgroup. \square

Theorem 3.7. *Let G act properly and cocompactly on a $\text{CAT}(0)$ cube complex \tilde{X} . Let A be an abelian subgroup of G . Then A has bounded packing in G .*

Proof. By Proposition 3.5 and the assumption that \tilde{X} is finite dimensional we can find a highest virtually free-abelian group A' that contains a finite index subgroup of A . The result now follows by combining Lemmas 3.1, 3.4, and Theorem 3.6. \square

4. SUBPRODUCT INTERSECTIONS

This section illustrates the following consequence of Theorem 3.6:

Theorem 4.1. *Let G act properly and cocompactly on a $CAT(0)$ cube complex \tilde{X} . Let $A \leq G$ be a highest free-abelian subgroup, and let $p = \text{rank}(A)$. There is a set $S = \{\hat{a}_1, \dots, \hat{a}_p\} \subseteq A$ such that the following holds: For any highest free-abelian subgroup $A' \leq G$, the intersection $A' \cap A$ is commensurable to a subgroup generated by a subset of S .*

Proving Theorem 4.1 requires the following consequence of the flat torus theorem.

Lemma 4.2. *Let A be a rank p virtually abelian group acting properly and cocompactly on a $CAT(0)$ cube complex $\prod_{i=1}^p C_i$, where each C_i is a quasiline. Then there exists a finite index free-abelian subgroup $\hat{A} \leq A$ with basis $\{\hat{a}_1, \dots, \hat{a}_p\}$ such that $\hat{a}_i \cdot (c_1, \dots, c_i, \dots, c_p) = (c_1, \dots, \hat{a}_i \cdot c_i, \dots, c_p)$ for each i .*

Proof. The action of A on $\prod_{i=1}^p C_i$ permutes the factors in the product, yielding a homomorphism $A \rightarrow S_p$ to the degree p symmetric group. Its kernel is a finite index subgroup $B \leq A$ such that the B -action on $\prod_{i=1}^p C_i$ is the product of B -actions on the factors. For each i there is a finite index subgroup $B_i \leq B$ that acts by translations on an invariant line $\ell_i \subset C_i$. Let $\hat{A} = \bigcap_{i=1}^p B_i$. Consider a homomorphism $\phi : \hat{A} \rightarrow \mathbb{Z}^p$ induced by the action of \hat{A} on $\prod_{i=1}^p \ell_i$. Since \hat{A} acts cocompactly on $\prod_{i=1}^p \ell_i$ we deduce that $[\mathbb{Z}^p : \phi(\hat{A})] < \infty$. Therefore, there are $\hat{a}_i \in \hat{A}$ such that $\phi(\hat{a}_i) = (0, \dots, 0, m_i, 0, \dots, 0)$, where $m_i \neq 0$ is the i -th entry. \square

We earlier defined the halfspaces $\overleftarrow{H}, \overrightarrow{H}$ associated to a hyperplane H of X to be the smallest subcomplexes containing the components $X - H$. The *small halfspaces* are the largest subcomplexes contained in the two components of $X - H$. Equivalently, the small halfspaces are the components of $X - N^o(H)$, where $N^o(H)$ is the union of open cubes intersecting H . Note that each small halfspace is convex as each component of $\partial N^o(H)$ is convex. It is readily verified that a subcomplex of X is convex if and only if it is the intersection of small halfspaces.

Lemma 4.3. *Let $\tilde{X} = \prod \tilde{X}_i$ where each \tilde{X}_i is a connected $CAT(0)$ cube complex. Then a convex subcomplex $\tilde{Y} \subseteq \tilde{X}$ is a product $\tilde{Y} = \prod \tilde{Y}_i$, where $\tilde{Y}_i \subseteq \tilde{X}_i$ is a convex subcomplex.*

Proof. Let \tilde{Y} be a convex subcomplex of \tilde{X} . Each 1-cube is the product of some 0-cubes and a single 1-cube in some factor \tilde{X}_i . An \tilde{X}_i hyperplane is a hyperplane which is dual to a 1-cube arising from a factor \tilde{X}_i . Let \tilde{Y}_i be the intersection of all small halfspaces containing \tilde{Y} that are associated to \tilde{X}_i hyperplanes. Then it is immediate that $\tilde{Y} = \prod \tilde{Y}_i$. \square

Proof of Theorem 4.1. By Theorem 3.6, A acts properly and cocompactly on a convex subcomplex $\tilde{Y} \cong \prod_{i=1}^p C_i \subseteq \tilde{X}$. A halfspace is *shallow* if it lies in a finite neighborhood of its hyperplane, and is *deep* otherwise. By passing to a smallest nonempty convex A -invariant subcomplex of \tilde{Y} , we may assume that no hyperplane in \tilde{Y} has both a shallow and a deep halfspace. The convex hull of an A -invariant p -flat $F \subseteq \tilde{Y}$ has this property. Indeed, each hyperplane intersecting F in a $(p-1)$ -flat necessarily has a pair of deep halfspaces, and a hyperplane containing F has two shallow halfspaces by cocompactness. By Lemma 4.2, there is a finite index subgroup $\hat{A} = \prod_{i=1}^p \langle \hat{a}_i \rangle$ of A such that $\langle \hat{a}_i \rangle$ acts cocompactly on C_i , and trivially on C_j for $j \neq i$. Let $S = \{\hat{a}_1, \dots, \hat{a}_p\}$. Similarly, A' cocompactly stabilizes a convex subcomplex $\tilde{Y}' \subseteq \tilde{X}$ which has its own induced product decomposition, and there exists a corresponding finite index subgroup $\hat{A}' = \prod_{i=1}^{p'} \langle \hat{a}'_i \rangle$ that acts cocompactly on \tilde{Y}' .

By Lemma 3.3 for each r there exists a cubical r -thickening $(\tilde{Y}')^{+r}$ containing $\mathcal{N}_r(\tilde{Y}')$ and $(\tilde{Y}')^{+r}$ is convex and \hat{A}' -cocompact. Choose r so that $\tilde{Y} \cap (\tilde{Y}')^{+r} \neq \emptyset$ and note that $\tilde{Y} \cap (\tilde{Y}')^{+r}$ is also convex. Therefore, by Lemma 4.3, the intersection is a subproduct $\tilde{Y} \cap (\tilde{Y}')^{+r} \subseteq \prod D_i \subseteq \prod C_i$ where each $D_i \subset C_i$ is a convex subcomplex. Thus each factor is either a quasiline, a quasiray, or a compact convex subcomplex. Furthermore, the action of $\hat{A} \cap \hat{A}'$ on $\tilde{Y} \cap (\tilde{Y}')^{+r}$ is cocompact. Indeed, the intersection $\tilde{Y} \cap (\tilde{Y}')^{+r}$ is the universal cover of a component of the fiber product of $\hat{A} \backslash \tilde{Y} \rightarrow G \backslash \tilde{X}$ and $\hat{A}' \backslash (\tilde{Y}')^{+r} \rightarrow G \backslash \tilde{X}$.

For each i , if D_i is a quasiline or compact then let $E_i = D_i$, and otherwise let E_i be the compact, $\hat{A} \cap \hat{A}'$ -invariant subcomplex contained in the intersection of all shallow halfspaces of D_i that have deep complements. Note that D_i is nonempty since by finite dimensionality, D_i is the intersection of finitely many shallow halfspaces whose associated hyperplanes intersect, and thus the Helly property implies the intersection is nonempty. Let $E = \prod E_i$. If E_i is a quasiline, then $\text{Stabilizer}_{\hat{A}}(E_i) = \langle \hat{a}_1, \dots, \hat{a}_i^{n_i}, \dots, \hat{a}_p \rangle$ for some $n_i > 0$ since $\hat{A} \cap \hat{A}'$ must act cocompactly on E . Otherwise, if E_i is compact, then $\text{Stabilizer}_{\hat{A}}(E_i) = \langle \hat{a}_1, \dots, \hat{a}_{i-1}, \hat{a}_{i+1}, \dots, \hat{a}_p \rangle$. Let $S_o \subseteq S$ be the subset of S such that $i \in S_o$ if E_i is a quasiline. Therefore $\text{Stabilizer}_{\hat{A}}(E)$ acts cocompactly on E , is commensurable to the subgroup generated by S_o , and contains $\hat{A} \cap \hat{A}'$.

Assume now that r is large enough that $(\tilde{Y})^{+r} \cap \tilde{Y}' \neq \emptyset$ and as before $(\tilde{Y})^{+r} \cap \tilde{Y}'$ contains a convex subcomplex of the form $E' = \prod E'_j$ where each E'_j is either a quasiline or compact, and $\text{Stabilizer}_{\hat{A}'}(E')$ acts cocompactly on E' and contains $\hat{A} \cap \hat{A}'$.

Any quasiline in E provides a bi-infinite sequence of nested hyperplanes. Every hyperplane in this sequence intersects \tilde{Y}' . Indeed, if some hyperplane in the sequence intersects $(\tilde{Y}')^{+r}$ but does not intersect \tilde{Y}' , then one side of the sequence would yield hyperplanes arbitrarily far from \tilde{Y}' , and this contradicts that $(\tilde{Y}')^{+r}$ lies within a uniform distance of \tilde{Y}' . We deduce that this quasiline corresponds to an entire quasiline of \tilde{Y}' and thus a quasiline of the subproduct E' .

Let E'' denote the subcomplex of $(\tilde{Y})^{+r} \cap (\tilde{Y}')^{+r}$ obtained by intersecting it with all halfspaces that contain $E \cup E'$. We now show that $E'' \subset \mathcal{N}_s(E)$ and $E'' \subset \mathcal{N}_s(E')$ for some $s > 0$. Indeed, suppose $E'' \not\subset \mathcal{N}_s(E)$ for each $s \geq 0$. Then for each s , there is a length s geodesic γ_s in E'' that starts at a 0-cube of E , and such that no hyperplane of E intersects γ_s . Let $\{H_{si}\}_{i=1}^s$ denote the sequence of hyperplanes dual to γ_s and let \vec{H}_{si} denote the halfspaces containing E . By definition of E'' , each H_{si} either intersects E' or separates E, E' . Note that the number of hyperplanes separating E, E' equals $d(E, E')$. Thus for each s , all but $d(E, E')$ of the hyperplanes in $\{H_{si}\}_{i=1}^s$ intersect E' . By finite dimensionality there is an upper bound on the number of pairwise crossing hyperplanes, and so by Ramsey's theorem, for each t there exists $S(t)$, such that γ_s is crossed by t pairwise disjoint hyperplanes whenever $s \geq S(t)$. We thus obtain arbitrarily long subsequence of hyperplanes that all intersect one of the finitely many factors of $E' = \prod D'_i$. Since the factors of E' are either finite or quasilines, we see that such a subsequence belongs to a quasiline of E' . Thus it belonged to a quasiline of E , as explained earlier. But all hyperplanes of a quasiline of E must cross E , which contradicts that no H_{si} crosses E .

We now show that $B = \text{Stabilizer}_{\hat{A}}(E)$ and $B' = \text{Stabilizer}_{\hat{A}'}(E')$ are commensurable within G . We have already shown that E' and E are coarsely equal, since each is coarsely equal to E'' . Let Υ denote the Cayley graph of G with respect to a finite generating set. A G -equivariant map $\Upsilon \rightarrow \tilde{X}$ shows that B, B' lie within finite neighborhoods of each other within Υ . The right action of B thus stabilizes a finite collection of right cosets of B' , and so B, B' are commensurable.

Let $H = B \cap B'$ which is a finite index subgroup of both B and B' . As $\hat{A} \cap \hat{A}' \leq B \leq \hat{A}$ and $\hat{A} \cap \hat{A}' \leq B' \leq \hat{A}'$, we have $H = \hat{A} \cap \hat{A}'$. Thus $\hat{A} \cap \hat{A}'$ is a finite index subgroup of B , hence acts cocompactly on E . The claim then follows from the fact that B is commensurable with a subgroup generated by S_o , and that $A \cap A'$ is commensurable to $\hat{A} \cap \hat{A}'$. \square

A \mathbb{Z}^p subgroup with a chosen product structure has $\binom{p}{q}$ distinct commensurability classes of \mathbb{Z}^q factor subgroups. We thus have the following corollary to Theorem 4.1:

Corollary 4.4. *Suppose G contains a highest free-abelian subgroup $A \cong \mathbb{Z}^p$. Suppose there are $\binom{p}{k} + 1$ other highest free-abelian subgroups $A_1, \dots, A_{\binom{p}{k}+1}$ such that the subgroups $A \cap A_i$ are pairwise non-commensurable and isomorphic to \mathbb{Z}^k . Then G cannot act properly and cocompactly on a $\text{CAT}(0)$ cube complex.*

We now illustrate Corollary 4.4 in a few situations.

Example 4.5. We describe an easy example of a group that acts properly on a finite dimensional $\text{CAT}(0)$ cube complex but does not have a finite index subgroup that acts properly and cocompactly on a $\text{CAT}(0)$ cube complex. Consider the group G presented as follows:

$$G = \langle a, b, r, s, t \mid [a, b], [a, r], [b, s], [ab, t] \rangle$$

Regard G as a multiple HNN extension of $\langle a, b \rangle$ with stable letters r, s, t , we see that G is a “tubular group”, and deduce that G acts properly on a finite dimensional $\text{CAT}(0)$ cube complex by utilizing the *equitable set* $\{a, b\}$ (see [Wis14] and

[Woo15]). However, G does not have a finite index subgroup G' that acts properly and cocompactly on a CAT(0) cube complex. Indeed, consider the following highest free-abelian subgroups: $A = \langle a, b \rangle$, $R = \langle a, r \rangle$, $S = \langle b, s \rangle$ and $T = \langle ab, t \rangle$. The intersections $R \cap A$, $S \cap A$, and $T \cap A$ are three pairwise non-commensurable cyclic subgroups of A , contradicting Corollary 4.4.

Note that G is a central HNN extension of the 2-dimensional right-angled Artin group $\langle a, b, r, s \mid [a, b], [a, r], [b, s] \rangle$, and so the virtually compact version of Theorem 5.5 fails without the assumption that H is highest.

Example 4.6. Let $\{\langle b_1 \rangle, \dots, \langle b_r \rangle, \langle c_1 \rangle, \dots, \langle c_r \rangle\}$ be a collection of pairwise incommensurable infinite cyclic subgroups of \mathbb{Z}^p , and suppose that $r > \frac{p}{2}$. Let G be the following multiple HNN extension of $\mathbb{Z}^p = \langle a_1, \dots, a_r \rangle$:

$$G = \langle a_1, \dots, a_p, t_1, \dots, t_r \mid [a_i, a_j] = 1, b_k^{t_k} = c_k : 1 \leq k \leq r \rangle$$

Then G does not contain a finite index subgroup that acts properly and cocompactly on a CAT(0) cube complex. Indeed, the subgroups $(\mathbb{Z}^p)^{t_i^{\pm 1}}$ intersect \mathbb{Z}^p in the various subgroups $\{\langle a_i \rangle, \langle b_i \rangle\}$ and so Corollary 4.4 applies.

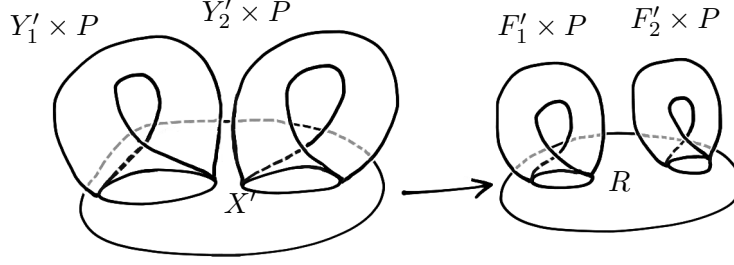
5. CENTRAL HNN EXTENSIONS OF MAXIMAL FREE-ABELIAN SUBGROUPS ARE SPECIAL

This section presumes familiarity with the notions of specialness and canonical completion and retraction. We refer to [HW08].

Lemma 5.1. *Let X be a virtually special cube complex. Let $f : Y \rightarrow X$ be a local isometry where Y is a compact nonpositively curved cube complex. Let (P, p) be a based graph. Let $Z = (X \sqcup (Y \times P)) / \{(y, p) \sim f(y) : \forall y \in Y\}$. Then Z is virtually special. Moreover, there is a finite special cover $\widehat{Z} \rightarrow Z$ such that the preimage of X is connected.*

Proof. Let $X' \rightarrow X$ be a finite degree special cover of X . Let $Y'_i \rightarrow X'$ be the finitely many elevations of $Y \rightarrow X$. For each i , let $C(Y'_i \rightarrow X')$ be the canonical completion of $Y'_i \rightarrow X'$ and identify Y'_i with its image in $C(Y'_i \rightarrow X')$. The canonical retraction $C(Y'_i \rightarrow X') \rightarrow Y'_i$ ensures that the maps $Y'_i \rightarrow C(Y'_i \rightarrow X')$ are *tidy* in the sense that they are injective and that no hyperplane U in $C(Y'_i \rightarrow X')$ *interosculates* with Y'_i in the sense that U is dual to an edge in Y'_i and is also dual to an edge that is not in Y'_i but has an endpoint in Y'_i . Suppose that a hyperplane U intersecting Y'_i were dual to an edge e not in Y'_i , but adjacent to a vertex $v \in Y'_i$. Then v must be adjacent to another edge e' in Y'_i dual to U since the retraction sends hyperplanes to hyperplanes. This implies a contradiction since the retraction must preserve the orientations of the dual edges, but U cannot self-oscillate.

Let \widehat{X} be a finite degree regular cover of X that factors through each $C(Y'_i \rightarrow X')$. Observe that now all elevations of Y to \widehat{X} are tidy, since tidiness is stable under covers. Finally, for each elevation $\widehat{Y}_j \hookrightarrow \widehat{X}$ of $Y \rightarrow X$, we adjoin a copy of $\widehat{Y}_j \times P$. We thus obtain a cover $\widehat{Z} \rightarrow Z$. The specialness of \widehat{Z} holds due to the tidy embeddings and a case-by-case analysis of its hyperplanes: each hyperplane $W \subset \widehat{X}$ has $(\widehat{Y}_j \cap W) \times P$ attached for each \widehat{Y}_j . Therefore no self-crossings, 1-sided hyperplanes, and


 FIGURE 2. The local isometry $Z \rightarrow S$.

no self-oscultations are introduced. The tidiness of each $\mathbb{C}(\hat{Y}_j \rightarrow \hat{X})$ guarantees that the new hyperplanes dual to the P factors cannot interoscultate with any hyperplane in \hat{X} , as each elevation factors through some Y'_i . \square

Remark 5.2. Lemma 5.1 can be generalised from the case where P is a graph, to the case where P is a special cube complex.

We will need the following technical result about right-angled Artin groups. A subgroup $A \leq G$ is *isolated* if $g^p \in A$ implies that $g \in A$ for some $p \in \mathbb{Z}$.

Lemma 5.3. *Let M be an abelian subgroup of a right-angled Artin group R . Suppose that M is not properly contained in another abelian subgroup, then M is isolated.*

Proof. Right-angled Artin groups are biorderable [DT92], therefore if $[g^p, h] = 1$ then $[g, h] = 1$. Indeed, if $ghg^{-1} > h$ then $(ghg^{-1})^n > h^n$ for all n , and likewise for $ghg^{-1} < h$. We conclude that by maximality of M , if $g^p \in M$, then $g \in M$. \square

Corollary 5.4. *If M is a maximal rank abelian subgroup of a right angled Artin group R , then it is a highest subgroup of R .*

Proof. If M is virtually contained in a higher rank subgroup M' of R , then there exists $g \in M - M'$ with $g^p \in M'$. This contradicts the isolation of M , by Lemma 5.3. \square

Theorem 5.5. *Let H be a finitely generated virtually [compact] special group. Let $A \subset H$ be a highest abelian subgroup. Let $G = H *_{A^t=A}$ be the HNN extension, where t is the stable letter commuting with A , then G is virtually [compact] special.*

Proof. Let X' be a [compact] nonpositively curved special cube complex such that $\pi_1 X'$ is isomorphic to a finite index subgroup of H . We may assume that X' has finitely many hyperplanes since H is finitely generated. Consider the local isometry to the associated Salvetti complex $X' \looparrowright R$, and note that $R = R(X')$ is compact since X' has finitely many hyperplanes.

Let $\{g_i\}$ be a finite set of representatives of the double cosets $\{Ag\pi_1 X'\}$. Let $\{A_i\}$ be the finitely many distinct intersections $\pi_1 X' \cap g_i^{-1} A g_i$. Each A_i is highest in H' , since A is highest in H . The subgroup $A_i \hookrightarrow \pi_1 R$ is contained in a maximal free-abelian group $\dot{B}_i \leq \pi_1 R$, which is highest in $\pi_1 R$ by Corollary 5.4. As A_i is highest in H' we have $[H' \cap \dot{B}_i : A_i] < \infty$. The quotient $p_i : \dot{B}_i / A_i \cong T \oplus \mathbb{Z}^m$ where $|T| < \infty$. The finite index subgroup $B_i = p_i^{-1}(\mathbb{Z}^m)$ of \dot{B}_i is still highest in $\pi_1 R$ and has the additional property that $H' \cap B_i = A_i$.

By Theorem 3.6, for each i there exists a local isometry $F_i \rightarrow R$ with F_i a compact nonpositively curved cube complex, such that $\pi_1 F_i$ maps to B_i . For each i , let $Y'_i \rightarrow R$ be the fiber-product of $X' \rightarrow R$ and $F_i \rightarrow R$. Note that by possibly replacing F_i with a sufficient convex finite thickening as provided by Lemma 3.3, we can assume that Y'_i is nonempty, so that $\pi_1 Y'_i = A_i$. Let $Z = X' \cup \bigcup (Y'_i \times P) / \sim$. Note that $\pi_1 Z$ is isomorphic to a finite index subgroup of G since the graph of groups for $\pi_1 Z$ covers the graph of groups of G . Let $S = R \cup \bigcup (F_i \times P) / \sim$ be the space obtained from R by attaching the various $F_i \times P$ along $F_i \times \{a\}$ using the map $F_i \rightarrow R$. See Figure 2.

A multiple use of Lemma 5.1 shows that S is virtually special. There is a local isometry $Z \rightarrow S$ given by the local isometry of X' into R extended along the local isometry $Y'_i \times P \rightarrow F_i \times P$, and hence Z is virtually special. \square

REFERENCES

- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999.
- [DT92] Gérard Duchamp and Jean-Yves Thibon. Simple orderings for free partially commutative groups. *Internat. J. Algebra Comput.*, 2(3):351–355, 1992.
- [HW] G. Christopher Hruska and Daniel T. Wise. Finiteness properties of cubulated groups. *Compositio Mathematica*. pp. 1–58, to appear.
- [HW08] Frédéric Haglund and Daniel T. Wise. Special cube complexes. *Geom. Funct. Anal.*, 17(5):1551–1620, 2008.
- [HW09] G. Christopher Hruska and Daniel T. Wise. Packing subgroups in relatively hyperbolic groups. *Geom. Topol.*, 13(4):1945–1988, 2009.
- [Rat94] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.
- [Rol98] M.A. Roller. Poc sets, median algebras and group actions. An extended study of Dunwoody’s construction and Sageev’s theorem. Preprint, Univ. of Southampton, 1998.
- [Sag95] Michah Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc. (3)*, 71(3):585–617, 1995.
- [Wis] Daniel T. Wise. The structure of groups with a quasiconvex hierarchy. Available at <http://www.math.mcgill.ca/wise/papers>. pp. 1–189. Submitted.
- [Wis14] Daniel T. Wise. Cubular tubular groups. *Trans. Amer. Math. Soc.*, 366(10):5503–5521, 2014.
- [Woo15] Daniel J. Woodhouse. Classifying finite dimensional cubulations of tubular groups. *arXiv:1502.02619*, pages 1–22, 2015. Submitted.

DEPT. OF MATH. & STATS., MCGILL UNIV., MONTREAL, QC, CANADA H3A 0B9

E-mail address: wise@math.mcgill.ca

E-mail address: daniel.woodhouse@mail.mcgill.ca